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Infinite-Scale Percolation in a New Type of Branching Diffusion Process

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We give an account and (basically) a solution of a new class of problems synthesizing percolation theory and branching diffusion processes. They lead to a novel type of stochastic process, namely branching processes with diffusion on the space of parameters distinguishing the branching "particles" from each other.

KEY WORDS: Branching diffusion, multiscale percolation.

1. INTRODUCTION

Branching stochastic processes[(1)] have always been very interesting for mathematicians and physicists. They describe well a multitude of phenomena, from chain reactions to population dynamics. On the other hand, ordinary[(2)] and multiscale[(3)] percolation play a crucial role in many applications.[(4)] In ref. 5 (see also ref. 6), on the basis of the idea of recoding,[(7)] necessary and sufficient conditions for the discrete hierarchical (multiscale) model of fracture (percolation of defects) were obtained.

In this paper we give an account and (basically) a solution of a new class of problems synthesizing percolation theory and branching diffusion processes. Such problems arise naturally in recent investigations of the global geometry of the inflationary early universe (ref. 8 and references therein).

In Section 2 we present a general formulation of the problem of infinite-scale percolation in a new type of branching diffusion process. The novelty is the diffusion not in the real space where branching "particles"

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live, but in the space of parameters distinguishing the "particles" from each other.

We do not specify in the following whether the A diffusion on a d-dimensional manifold, which we assume from the very beginning, corresponds to some stochastic differential equation (SDE) in the sense of Itô or in the sense of Stratonovich. In the applications of ref. 8 we use Stratonovich's symmetrized calculus because white noise there is the limit of a colored noise with small time correlation. The generating operator ($\hat{A}$ operator) of the diffusion in the local coordinate frame is given as follows ($i, j, k = 1, 2,..., d$; the summation over repeated indexes is assumed):

(i) After Itô

$$\hat{A}_I f(X) = \frac{1}{2} \sigma^{ik}(X) \sigma^{kj}(X) \frac{\partial^2}{\partial X^i \partial X^j} f(X) + \mathscr{F}^i(X) \frac{\partial}{\partial X^i} f(X) \tag{1}$$

(ii) After Stratonovich

$$\hat{A}_S f(X) = \frac{1}{2} \sigma^{ik}(X) \frac{\partial}{\partial X^i} \left(\sigma^{kj}(X) \frac{\partial}{\partial X^j} f(X) \right) + \mathscr{F}^i(X) \frac{\partial}{\partial X^i} f(X) \tag{2}$$

It corresponds to the SDE (in the proper sense)

$$dX_t^i = \mathscr{F}^i(X)\, dt + \sigma^{ik}(X) \circ dW_t^k \tag{3}$$

Here $X_t^i \in R^d$, and W_t^k is the d-dimensional Wiener process. We add to the random walk (3) a branching with intensity $n(X)$ [i.e., the probability density of branching of the "particle" at X during the time Δt is $n(X)\, \Delta t$]. Then we relate to such a branching diffusion some process of breaking up of D-dimensional cubes and their coloring and obtain an infinite-scale (in the $t \to \infty$ limit) percolation problem.

After some preliminaries (Section 3), the solution of the problem is stated in Section 4, where necessary and sufficient conditions for percolation are obtained. In Section 5 we illustrate our method in the case of the simplest model. In the summary in Section 6 possible generalizations are discussed.

2. THE PROBLEM

Let the basic d-dimensional diffusion process generated by the operator $\hat{A}$ of (1), (2) be given by

$$\lim_{\Delta t \downarrow 0} \mathsf{E} \left\{ \frac{f(X_{t+\Delta t}) - f(X_t)}{\Delta t} \,\middle|\, X_t = X \right\} = \hat{A} f(X) \tag{4}$$

where $X_t \in \mathscr{M}$, with $\mathscr{M}$ a d-dimensional manifold. The action of $\hat{A}$ is endowed by the boundary condition

$$f(X)|_{X \in \Gamma} = 0 \tag{5}$$

where $\Gamma \subset \mathscr{M}$ is a closed subset of $\mathscr{M}$ (an absorbing boundary).

On $\mathscr{M} \backslash \Gamma$ the smooth functions $n(X)$, $\mathscr{F}^i(X)$, and $\sigma^{ij}(X)$ are defined [see (3)], where $n(X)$ and $\sigma^{ij}(X)$ are assumed to be positive, such that Γ is accessible. Let there appear at each branching instead of one "particle" at point X, the given number r of "particles" at that point, which continue to evolve as branching diffusion processes independent of each other. When a "particle" reaches the absorbing boundary Γ, it stays there forever without branching.

Now we will associate this branching diffusion with the following picture. Let at $t=0$ a D dimensional cubic net be given, consisting of cubes of unit size. With each cube of this net independently we set in correspondence a random point $X_0 \in \mathscr{M}$ with the probability density $\gamma(X_0)$. The subsequent evolution of each cube is independent of the rest of the net. This evolution is determined by the above-described basic branching diffusion on $\mathscr{M}$. At $t=0$, r trajectories $X_t^{(i)}$ ($i=1,..., r$) of random walkers (3) start at X_0. We divide the unit cube (call it the cube of zero level) into r smaller equal cubes of the first level (we assume that $r=k^D$, so that the cubes of the first level are k times smaller than the unit cube). Each first-level cube is set in correspondence with one of the points $X_1^{(i)}$—the points where the ith "particle" branches for the first time. If the "particle" was absorbed at Γ, we attribute to the corresponding first-level cube the proper point $X_1 \in \Gamma$, color it black, and leave it in peace. We divide the other cubes, which should not be colored this time (we call them "white" or "living" cubes), once more into the $r=k^D$ cubes of the second level and do the same procedure, starting from $X_1^{(i)} \in \mathscr{M} \backslash \Gamma$. Thus, coloring some cubes at each level q (when $X_q^{(i)} \in \Gamma$) and dividing white cubes further, we will obtain some infinite-scale (provided that the process does not degenerate at a finite level) picture of black cubes of different sizes in a "sea" of living white cubes (see Fig. 1).

Denote by $\mathscr{B}_t$ the multitude of all black cubes of all sizes (and by $\mathscr{W}_t$ that of all white cubes) belonging to a particular unit cube at the time t. We assume that two cubes of (possibly) different sizes are connected if they have common face. Now we are ready to formulate the problem.

Problem. Consider the whole net of (zero-level) cubes and denote by B_t the union of all the multitudes $\mathscr{B}_t$ belonging to all unit cubes in the original cubic net. Let $\gamma(X)$, $n(X)$, and $\hat{A}$ be given. Does B_t percolate in the

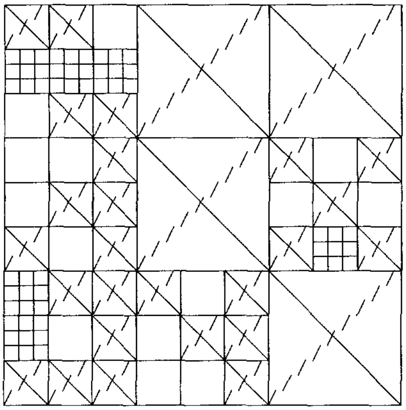

Fig. 1. A part of the net containing black squares of different sizes in a sea of white squares. Crossed squares are black.

$t \to \infty$ limit (we will denote the corresponding multitudes of cubes as $\mathscr{B}_\infty$ and B_∞, respectively)?

Note. Percolation of B_∞ means that a connected non-self-intersecting path a.s. exists on B_∞ from some point (call it the origin of the coordinate frame) up to infinity.

3. PRELIMINARIES

First of all, let us elucidate the condition of nondegeneracy of the process of breaking up of the cubes. Let $\mu(t, V)$ be the number of random walkers within the region $V \subset \mathscr{M} \backslash \Gamma$ at the time t. Introduce the generating function

$$u(t, X, z) = \mathsf{E}\{z^{\mu(t, V)} \mid X_0 = X\} \tag{6}$$

Noting that the evolution of $u(t, X, z)$ is driven by the diffusion of particles and by the branching processes, we obtain

$$\begin{aligned} u(t + \Delta t, X, z) = {} & [1 - n(X)\, \Delta t]\, \mathsf{E}\{u(t, X_{\Delta t}, z) \mid X_0 = X\} \\ & + n(\Delta)\, \Delta t(u(t, X, z))^r \end{aligned} \tag{7}$$

Here we considered the whole tree of the branching diffusion process from $t = 0$ to $t + \Delta t$ and divided it into two parts—a part from $t = 0$ to $t = \Delta t$ and a part from $t = \Delta t$ to $t + \Delta t$. Up to the first order in Δt, we have only two mutually exclusive possibilities of evolution of the process from $t = 0$ to $t = \Delta t$.

1. The single original particle does not branch during this time and it diffuses to a new place $X_{\Delta t}$. The tree of the branching diffusion process from $t=\Delta t$ to $t+\Delta t$ differs from the one corresponding to the interval of time $[0, t]$ only by its origin $X_{\Delta t}$. This possibility is represented by the first term in the rhs of Eq. (7).

2. The original particle branches just one time during the period from $t=0$ to $t=\Delta t$. There appear r species of the tree of the branching diffusion process and, in the first order in Δt, we should not distinguish them from the tree corresponding to the interval $[0, t]$. This possibility is represented by the second term in the rhs of Eq. (7).

Then, using Eq. (4), one easily derives from (7) the (backward) differential equation for the generating function

$$\frac{\partial}{\partial t} u(t, X, z) = \hat{A} u(t, X, z) + n(X)[u^r(t, X, z) - u(t, X, z)] \tag{8}$$

and the boundary and initial conditions

$$u(t, X, z)|_{X \in \Gamma} = 1 \tag{9}$$

$$u(0, X, z) = \begin{cases} z & \text{if } \quad X \in V \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

As a consequence of the definition (6), we can write down the equations for the (factorial) moments of $\mu(t, V)$, differentiating Eqs. (8)–(10). Note that

$$m!_l(t, X) = \mathsf{E}\{\mu(\mu-1)\cdots(\mu-l) \mid X_0 = X\} = \frac{\partial^l}{\partial z^l} u(t, X, z)|_{z=1} \tag{11}$$

and

$$m_l(t, X) = \mathsf{E}\{\mu^l(t, V) \mid X_0 = X\} = \left(z \frac{\partial}{\partial z}\right)^l u(t, X, z)|_{z=1} \tag{12}$$

In particular, the equation for the first moment

$$m_1(t, X) = z \frac{\partial}{\partial z} u(t, X, z)|_{z=1}$$

(the average number of random walkers in V at the time t) is

$$\frac{\partial}{\partial t} m_1(t, X) = \hat{A} m_1(t, X) + (r-1)\, n(X)\, m_1(t, X) \tag{13}$$

$$m_1(t, X)|_{X \in \Gamma} = 0 \tag{14}$$

$$m_1(0, X) = \begin{cases} 1 & \text{if} \quad X \in V \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Equations (13)–(15) have an asymptotic ($t \to \infty$) solution

$$m_1(t, X) \propto e^{\lambda_1 t} \psi_1(X) \int_V dY\, \pi_1(Y) \tag{16}$$

Here $\psi_1(X)$ is the unique strictly positive real eigenfunction of the marginal problem

$$\hat{A}\psi_1(X) + (r-1)\, n(X)\, \psi_1(X) = \lambda_1 \psi_1(X) \tag{17}$$

$$\psi_1(X)|_{X \in \Gamma} = 0 \tag{18}$$

and λ_1 is the corresponding (real) eigenvalue. The function $\pi_1(X)$ (invariant density) is the unique strictly positive eigenfunction of the adjoint equation with the same eigenvalue λ_1,

$$\hat{A}^{\dagger}\pi_1(X) + (r-1)\, n(X)\, \pi_1(X) = \lambda_1 \pi_1(X) \tag{19}$$

$$\pi_1(X)|_{X \in \Gamma} = 0 \tag{20}$$

The normalizations are as follows:

$$\int_{\mathscr{M} \setminus \Gamma} \pi_1(X)\, dX = 1; \qquad \int_{\mathscr{M} \setminus \Gamma} \pi_1(X)\, \psi_1(X)\, dX = 1 \tag{21}$$

One can see from (16) that if $\lambda_1 < 0$, then the branching process a.s. degenerates in the limit $t \to \infty$ (in our notations $\mathscr{W}_\infty = \varnothing$). If $\lambda_1 > 0$, then the branching process is supercritical and $\mathscr{W}_\infty \neq \varnothing$. We assume the latter case in the rest of the paper.

Let us introduce two other useful functions. The first one is the probability of the event $\aleph_t$ that the particle gets absorbed at Γ at some time less than or equal to t provided that it started to diffuse from some point $X \in \mathscr{M} \setminus \Gamma$ at the time $t = 0$ and did not branch before being absorbed:

$$\rho(X, t) = \mathsf{P}\{\aleph_t \mid X_0 = X\} \tag{22}$$

Consider the moment of time $t + \Delta t$. It is obvious from the definition above that (in the first order in Δt)

$$\rho(X, t + \Delta t) = [1 - n(X)\, \Delta t]\; \mathsf{E}\{\rho(X_{\Delta t}, t) \mid X_0 = X\} \tag{23}$$

and the following (backward) differential equation as well as the initial and boundary conditions follow immediately from (23) and (4):

$$\frac{\partial}{\partial t}\rho(X, t) = \hat{A}\rho(X, t) - n(X)\,\rho(X, t) \tag{24}$$

$$\rho(X, t)|_{X\in\Gamma} = 1 \tag{25}$$

$$\rho(X, 0) = \begin{cases} 1 & \text{if} \quad X \in \Gamma \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

In the rest of this paper we will use only the stationary probability $\rho(X)$ of absorption of a particle which starts its random walk at X without branching before absorption. It is given by the stationary solution of Eqs. (24)–(26). Since $n(X)$ is positive and the maximal eigenvalue of $\hat{A}$ with zero boundary condition on Γ is negative, $\lambda = 0$ is not an eigenvalue of $[\hat{A} - n(X)]$ and the stationary equation

$$\hat{A}\rho(X) - n(X)\,\rho(X) = 0; \qquad \rho(X)|_{X\in\Gamma} = 1 \tag{27}$$

has a unique solution. Of course, the relation $\lim_{t\to\infty} \rho(X, t) = \rho(X)$ is satisfied.

Then, let us introduce the probability density in Y

$$K(X, Y, t) = \mathsf{P}\{\Re_t(Y) \mid X_0 = X\} \tag{28}$$

where $\Re_t(Y)$ is the event that the particle branches for the first time in the infinitesimal volume $Y + dY$ at some time less than or equal to t, provided that it starts to diffuse at X at $t = 0$. Consider the evolution of this quantity after time Δt in the first order in Δt. One has

$$\begin{aligned} K(X, Y, t) = n(X)\,\Delta t\,\delta(Y - X) + [1 - n(X)\,\Delta t] \\ \times \mathsf{E}\{K(X_{\Delta t}, Y, t + \Delta t) \mid X_0 = X\} \end{aligned} \tag{29}$$

Here the first term represents the event that the particle branches during the time interval $[0, \Delta t]$ and we should not distinguish the position of the particle at $t = 0$ and $t = \Delta t$ in the first order in Δt. The second term represents the event that the particle does not branch during the time interval $[0, \Delta t]$ and in the first order in Δt the only difference we should take into account for that particle's trajectory during the time interval $[\Delta t, t + \Delta t]$ is its random initial point $X_{\Delta t}$ at the beginning of that interval. These are the only two mutually exclusive events which exist in the first order in Δt. From the definition (28) and Eq. (29) one easily obtains the (backward) differential equation and the boundary and initial conditions

$$\frac{\partial}{\partial t} K(X, Y, t) = \hat{A}_X K(X, Y, t) - n(X)\, K(X, Y, t) + n(Y)\, \delta(X - Y) \quad (30)$$

$$K(X, Y, t)|_{X \in \Gamma} = K(X, Y, t)|_{Y \in \Gamma} = 0 \quad (31)$$

$$K(X, Y, 0) = \delta(X - Y) \quad (32)$$

Here $\hat{A}_X$ denotes the generating differential operator of diffusion acting on the first variable of $K(X, Y, t)$. In what follows we will need only the stationary probability density $K(X, Y)$, which satisfies the equations

$$\hat{A}_X K(X, Y) - n(X)\, K(X, Y) + n(Y)\, \delta(X - Y) = 0 \quad (33)$$

$$K(X, Y)|_{X \in \Gamma} = K(X, Y)|_{Y \in \Gamma} = 0 \quad (34)$$

One can write down the expression for $K(X, Y)$ through the complete orthonormal set of eigenfunctions $K_s(X)$ and eigenvalues κ_s of the marginal problem

$$\hat{A} K_s(X) - n(X)\, K_s(X) = \kappa_s K_s(X) \quad (35)$$

$$K_s(X)|_{X \in \Gamma} = 0 \quad (36)$$

Recalling the following properties of the eigenfunctions

$$\int_{\mathscr{M} \setminus \Gamma} K_i(X)\, K_j(X)\, dX = \delta_{ij}; \qquad \sum_{s=1}^{\infty} K_s(X)\, K_s(Y) = \delta(X - Y) \quad (37)$$

one easily obtains

$$K(X, Y) = -n(Y) \sum_{s=1}^{\infty} \frac{1}{\kappa_s} K_s(X)\, K_s(Y) \quad (38)$$

Now, with the functions $\rho(X)$ and $K(X, Y)$ in hand, we are ready to solve the problem stated in the preceding section.

4. INFINITE-SCALE PERCOLATION

In this section we will find the percolation characteristics of B_∞ using the renormalization relations which arise as a result of the recoding procedure. This procedure was introduced in the case of the simpler model of discrete hierarchical fracture in ref. 7 and was investigated in detail in ref. 5.

Consider the multitude $\mathscr{B}_t$ belonging to a particular unit cube at some large time t. Let us concentrate for the moment on the percolation characteristics of this unit cube. Let us define the notion of "strongly defective" cubes.[4]

Definition 1. A cube of level L is called "strongly defective" of rank 0 ($\mathrm{SD}_L^{(0)}$) if it is black. It is called "strongly defective" of rank $m>0$ ($\mathrm{SD}_L^{(m)}$) if either it is black or the following two conditions are satisfied:

1. There exists a connected cluster of $\mathrm{SD}_{L+1}^{(m-1)}$ cubes of level $L+1$ which connects each pair of faces of the given cube.
2. Strictly more than half of the surface of each face of the cube is covered by $\mathrm{SD}_{L+1}^{(m-1)}$ cubes belonging to one of the clusters defined in 1.

A cube is said to be SD if it is $\mathrm{SD}^{(n)}$ for some $n \geqslant 0$.

We assume in the main body of the text $k \geqslant 3$. In the $k=2$ case more case is needed due to the fact that there are no nontrivial configurations satisfying condition 2 (see Section 6). The meaning of this definition is revealed by the following obvious result.

Proposition 1. If any number of SD cubes of some level L are attached face to face, then there is a connected cluster of black cubes (consisting, maybe, of cubes of different levels $K \geqslant L$) running through them. If a cube is $\mathrm{SD}^{(m)}$, then it is $\mathrm{SD}^{(n)}$ for every $n>m$.

Note that it is the condition 2 of Definition 1 that ensures the percolation of the black color through the attached faces of SD cubes. Thus, SD cubes are as good as black ones when we deal with the percolation of the black color through the net of cubes. It is this kind of substitution of black cubes by SD cubes that was called the "recoding procedure" in ref. 7 and that enables us to follow the "renormalization approach" of refs. 5 and 7.

The opposite proposition (that SD cubes always consist of SD cubes) is false (see Fig. 2). That is why we can obtain only the sufficient (but not necessary) condition for percolation assuming that the probability of the level-L cube to be SD can be computed neglecting such configurations as in Fig. 2,

$$\mathsf{P}_X\{\mathrm{SD}_L\} = \mathsf{P}_X\{B_L\} + (1-\mathsf{P}_X\{B_L\})\,\mathsf{P}_X\left\{\begin{matrix}\text{SD configuration of}\\ \text{level } L+1 \text{ cubes}\end{matrix}\right\} \tag{39}$$

where the subscript L denotes the level of the cubes and the subscript X takes into account the fact that we deal with probabilities depending on the

[4] We call black cubes "defects" to resemble the rock fracture theory.[7]

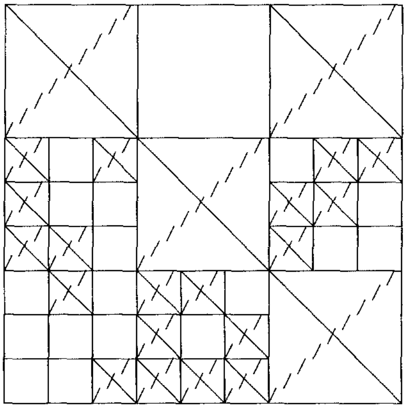

Fig. 2. Counterexample. The square of zero level is SD, although four of nine squares of the first level are not SD.

value of the parameter associated with the given cube. We are interested in calculation of $\mathsf{P}_X\{\mathrm{SD}_0\}$. Let us recall that SD means $\mathrm{SD}^{(n)}$ for some $n \geqslant 0$. In other words, we should find the limit

$$\mathsf{P}\{X\} = \lim_{n \to \infty} \mathsf{P}_X\{\mathrm{SD}_0^{(n)}\} \tag{40}$$

Let us forget for a while about the time dependence of the problem and use the stationary equation

$$\mathsf{P}_X\{\mathrm{SD}_0^{(n)}\} = \rho(X) + [1-\rho(X)] \int_{\mathscr{M}\setminus\Gamma} K(X, Y)\, Q_{D,k}(\mathsf{P}_Y\{\mathrm{SD}_1^{(n-1)}\})\, dY \tag{41}$$

where $\rho(X)$ and $K(X, Y)$ were introduced in the preceding section [see (27), (33)–(38)]. Equation (41) gives the probability for the zero-level cube to be $\mathrm{SD}_0^{(n)}$ in terms of the probabilities of the first-level cubes to be $\mathrm{SD}_1^{(n-1)}$ provided that the parameters associated with them are such that the "particle" corresponding to the zero-level cube is the "parent" of the particles corresponding to the first-level cubes.

The combinatorial function $Q_{D,k}(p)$ which appeared in (41) is a polynomial in p of rth degree. It counts the probability of SD configurations of level-$(L+1)$ cubes inside of cubes of level L (we assume for the moment that p is the probability of black color and $q = 1-p$ is the probability of white). We will consider in this paper only the simplest case of two-dimensional cubes (squares) and $k=3$, although it is only a straightforward combinatorial problem to find $Q_{D,k}(p)$ for the other cases,[5]

$$Q_{2,3}(p)=p^9+9p^8q+20p^7q^2=12p^9-31p^8+20p^7 \tag{42}$$

Equation (41) can be continued by the hierarchy

$$\begin{aligned} \mathsf{P}_X\{\mathrm{SD}_1^{(n-1)}\} &= \rho(X)+[1-\rho(X)]\int_{\mathcal{M}\setminus\Gamma} dY\, K(X,Y)\, Q_{D,k}(\mathsf{P}_Y\{\mathrm{SD}_2^{(n-2)}\}) \\ &\vdots \end{aligned} \tag{43}$$

$$\mathsf{P}_X\{\mathrm{SD}_{n-1}^{(1)}\} = \rho(X)+[1-\rho(X)]\int_{\mathcal{M}\setminus\Gamma} dY\, K(X,Y)\, Q_{D,k}(\mathsf{P}_Y\{\mathrm{SD}_n^{(0)}\}) \tag{44}$$

Then, recalling that

$$\mathsf{P}_X\{\mathrm{SD}_{\text{any level}}^{(0)}\}=\rho(X) \tag{45}$$

we easily find

$$\mathsf{P}_X\{\mathrm{SD}_0^{(n)}\}=\underbrace{F[F[\cdots F}_{n\text{ times}}[\rho(X)]\cdots]]=F_n[\rho(X)] \tag{46}$$

Here $F_n[\phi(X)]$ denotes the nth iteration of the integral operator

$$F[\phi(X)]=\rho(X)+[1-\rho(X)]\int_{\mathcal{M}\setminus\Gamma} dY\, K(X,Y)\, Q_{D,k}(\phi(Y)) \tag{47}$$

From Eqs. (41)–(47) one can derive that in the $n\to\infty$ $(t\to\infty)$ limit the iterations converge [under some restrictions on $\rho(X)$ and $K(X,Y)$ which are necessary to guarantee the uniqueness of the fixed point] to the solution of the following integral equation:

$$\mathsf{P}\{X\}=F[\mathsf{P}\{X\}] \tag{48}$$

It is worth noting that after Eqs. (41)–(44) it becomes clear why we can substitute a t-dependent percolation problem by n-dependent hierarchical one. Indeed, up to now we have not mentioned that not all "white" cubes of a given level break up simultaneously and consequently there exist many "white" cubes of different sizes at the time t. However, our approach

[5] One should remember, however, that computational difficulties grow exponentially fast with growing D or k, and the finding of the functions $Q_{D,k}(p)$, although remaining a finite combinatorial problem, becomes very cumbersome.

is insensitive to this difficulty because different cubes can join the "renormalization flow" at different scales, but all of them approach eventually the same fixed point (48). The only modification that can arise from this notion is the somewhat stronger condition on the limiting rate $t \to \infty$ when one proves the coincidence of $\lim_{t \to \infty} \mathsf{P}_X\{\mathrm{SD}_0(t)\}$ and $\mathsf{P}\{X\}$ from (40).

In some cases, when the full transition probability density $K(X, Y)$ of (38) can be approximated by a finite sum like

$$-n(Y) \sum_{s=1}^{J} \frac{1}{\kappa_s} K_s(X)\, K_s(Y) \tag{49}$$

the corresponding nonlinear integral equation (48) reduces to an algebraic one. This fact can be useful in applications.[8]

Suppose that we found the solution of (48). Then the solution of the percolation problem is straightforward. The quantity

$$p_* = \int_{\mathscr{M} \setminus \Gamma} \gamma(X_0) \mathsf{P}\{X_0\}\, dX_0 \tag{50}$$

has the meaning of the probability that the cube of zero level is SD. If $p_D^{(1)}$ is the percolation threshold for the site percolation problem on a cubic D-dimensional lattice Z^D, then, relating black sites to SD cubes, white sites to non-SD cubes, and recalling Proposition 1, we come immediately to the following result.

Theorem 1. The sufficient condition for percolation of B_∞ is

$$p_* > p_D^{(1)} \tag{51}$$

The necessary condition for percolation is the negation of the sufficient condition for nonpercolation. Let us introduce the notion of "strictly closing" (SC) configurations to obtain the sufficient condition for nonpercolation.

Definition 2. A cube of level L is called "strictly closing" of zero rank ($\mathrm{SC}_L^{(0)}$) if it is white. A cube is called "strictly closing" of rank m ($\mathrm{SC}_L^{(m)}$) if it is either white or it consists of such a configuration of $\mathrm{SC}_{L+1}^{(m-1)}$ cubes that there is no connected cluster of the remaining cubes of level $L+1$ belonging to the $\mathrm{SC}_L^{(m)}$ cube which could connect any pair of faces of that cube. A cube is said to be SC if it is $\mathrm{SC}^{(n)}$ for some $n \geqslant 0$.

One can easily verify the following result.

Proposition 2. Any number of SC cubes attached face to face do not contain any percolating path of black cubes. If the cube is $\mathrm{SC}^{(m)}$, then it is $\mathrm{SC}^{(n)}$ for every $n > m$.

Now it is clear that our "renormalization" procedure can be applied. Let us denote by a bar all quantities concerning the SC cubes. In complete analogy with Eqs. (39)–(48), we obtain

$$\bar{\mathsf{P}}_X\{\mathrm{SC}_0^{(n)}\} = [1-\rho(X)] \int_{\mathscr{M}\setminus\Gamma} dY\, K(X, Y)\, \bar{Q}_{D,k}(\bar{\mathsf{P}}_X\{\mathrm{SC}_1^{(n-1)}\}) \quad (52)$$

$$\vdots$$

$$\bar{\mathsf{P}}_X\{\mathrm{SC}_{n-1}^{(1)}\} = \underbrace{\Phi[\Phi[\cdots\Phi}_{n \text{ times}}[\bar{\mathsf{P}}_X\{\mathrm{SC}_n^{(0)}\}]\cdots]] = \Phi_n[1-\rho(X)] \quad (53)$$

where we used $\bar{\mathsf{P}}_X\{\mathrm{SC}_{\text{any level}}^{(0)}\} = 1-\rho(X)$ and the definitions

$$\Phi[\phi(X)] = [1-\rho(X)] \int_{\mathscr{M}\setminus\Gamma} dY\, K(X, Y)\, \bar{Q}_{D,k}(\phi(Y)) \quad (54)$$

$$\bar{\mathsf{P}}\{X\} = \lim_{n\to\infty} \bar{\mathsf{P}}_X\{\mathrm{SC}_0^{(n)}\} \quad (55)$$

Then, we have (under the condition that $\Phi[\phi(X)]$ has a unique fixed point)

$$\bar{\mathsf{P}}\{X\} = \Phi[\bar{\mathsf{P}}\{X\}] \quad (56)$$

Here $\bar{Q}_{D,k}(q)$ is the analog of $Q_{D,k}(p)$ for closing configurations ($q = 1-p$ is the probability of white color, p is the probability of black). We will use in the next section only the simplest combinatorial function of that type,

$$\bar{Q}_{2,3}(q) = q^9 + 5q^8p + 10q^7p^2 + 4q^6p^3 + q^5p^4 = 3q^9 - 7q^8 + 4q^7 + q^5 \quad (57)$$

$\bar{\mathsf{P}}\{X\}$ is the probability for the zero-level cube to be SC. Associating white sites of the black site percolation problem on Z^D to SC cubes, black sites to non-SC cubes, and keeping in mind Proposition 2, we come to the following result.

Theorem 2. Let $p_D^{(2)} = 1 - q_D^{(2)}$ be the nonpercolation threshold for the black site percolation problem on Z^D. Denote

$$q_* = \int_{\mathscr{M}\setminus\Gamma} \gamma(X_0)\, \bar{\mathsf{P}}\{X_0\}\, dX_0 \quad (58)$$

Then the necessary condition for percolation (negation of the sufficient condition for nonpercolation) of B_∞ is

$$q_* < q_D^{(2)} \quad (59)$$

5. THE SIMPLEST MODEL

Here we will investigate the simplest model of the type described above. It corresponds to homogeneous diffusion on the line segment $\mathscr{M}=[0;l]$, $\Gamma=\{0;1\}$, $d=1$, with

$$\mathscr{F}(x)=0; \qquad \sigma(x)=\sigma=\text{const} \tag{60}$$

$$n(x)=a^2=\text{const}; \qquad \gamma(x)=\frac{1}{l} \tag{61}$$

$$\hat{A}f(x)=\frac{\sigma^2}{2}\frac{d^2}{dx^2}f(x) \tag{62}$$

and to the $D=2$, $k=3$, $r=9$ case of the cubic net. From Eqs. (17) and (18) we obtain

$$\lambda_1=8a^2-\frac{\pi^2\sigma^2}{2l^2} \tag{63}$$

and the nondegeneracy condition is $al/\sigma>\pi/4$. Assuming this, one can derive from Eqs. (27) and (33)–(38)

$$\rho(z)=\frac{\cosh(\frac{1}{2}\zeta-z)}{\cosh(\frac{1}{2}\zeta)} \tag{64}$$

$$K(z,u)=\begin{cases}\dfrac{\zeta}{l\sinh(\zeta)}\sinh(\zeta-z)\sinh(u) & \text{if}\quad z\geqslant u\\[2ex] \dfrac{\zeta}{l\sinh(\zeta)}\sinh(\zeta-u)\sinh(z) & \text{if}\quad z\leqslant u\end{cases} \tag{65}$$

where we have introduced the dimensionless variables and the parameter ζ, which is in fact the only dimensionless parameter in this model and on which the behavior of the model depends essentially,

$$z=\frac{\sqrt{2}\,ax}{\sigma}; \qquad u=\frac{\sqrt{2}\,ay}{\sigma}; \qquad \zeta=\frac{\sqrt{2}\,al}{\sigma} \tag{66}$$

Equation (48) now reads

$$\begin{aligned}\mathsf{P}\{z\}=&\frac{\cosh(\frac{1}{2}\zeta-z)}{\cosh(\frac{1}{2}\zeta)}+\left(1-\frac{\cosh(\frac{1}{2}\zeta-z)}{\cosh(\frac{1}{2}\zeta)}\right)\\ &\times\left[\sinh(\zeta-z)\int_0^z du\,\frac{\sinh(u)}{\sinh(\zeta)}\,Q_{2,3}(\mathsf{P}\{u\})\right.\\ &\left.+\sinh(z)\int_z^\zeta du\,\frac{\sinh(\zeta-u)}{\sinh(\zeta)}\,Q_{2,3}(\mathsf{P}\{u\})\right]\end{aligned} \tag{67}$$

Introduce the function

$$\beta(z) = \int_0^z du \, \frac{\sinh(u)}{\sinh(\zeta)} \, Q_{2,3}(\mathsf{P}\{u\}) \tag{68}$$

Because we have $\rho(\zeta - z) = \rho(z)$, symmetry reasons lead us to $\mathsf{P}\{\zeta - z\} = \mathsf{P}\{z\}$ and Eq. (67) now reads

$$\mathsf{P}\{z\} = \frac{\cosh(\frac{1}{2}\zeta - z)}{\cosh(\frac{1}{2}\zeta)} + \left(1 - \frac{\cosh(\frac{1}{2}\zeta - z)}{\cosh(\frac{1}{2}\zeta)}\right) \times [\sinh(\zeta - z)\,\beta(z) + \sinh(z)\,\beta(\zeta - z)] \tag{69}$$

One can derive an ordinary differential equation for $\beta(z)$ from (68) and (69), but it is too involved and we do not produce it here. Instead, we present the result of the computer solution of inequality (51) in terms of ζ, which was calculated using (68) and (69) and assuming the known value of $p_2^{(1)} = 0.59$. Thus, the sufficient condition is

$$\zeta \leqslant \zeta_{\text{suff}} = 3.29 \tag{70}$$

Equations for the necessary condition are [see (54)–(59)]

$$\bar{\mathsf{P}}\{z\} = \left(1 - \frac{\cosh(\frac{1}{2}\zeta - z)}{\cosh(\frac{1}{2}\zeta)}\right) [\sinh(\zeta - z)\, g(z) + \sinh(z)\, g(\zeta - z)] \tag{71}$$

$$g(z) = \int_0^z du \, \frac{\sinh(u)}{\sinh(\zeta)} \, \bar{Q}_{2,3}(\bar{\mathsf{P}}\{u\}) \tag{72}$$

Our estimate for the necessary condition (59) is too crude for this particular model and we do not produce the result here. However, we think that the sufficient condition (70) is reasonably close to the exact critical value of ζ. This can be confirmed by calculation of the next level of corrections to it [compare with Eqs. (73), (74), where such corrections are the leading ones].

6. SUMMARY AND DISCUSSION

We have described a new class of problems, synthesizing problems from familiar percolation and branching diffusion. To find the percolation characteristics, one has to solve the analytic relations (47)–(51) and (54)–(59).

Let us emphasize some possible generalizations. First of all, it is worth noting that face-connectedness, which we assumed throughout this paper, can be replaced by a great variety of other "quasilocal" definitions of connectedness. By this we mean a definition which would deal with some finite clusters of black cubes (possibly of different sizes) with some rules restricting their configurations. If something like Propositions 1 and 2 holds, then only a minor modification of our approach, reduced mainly to a change of the combinatorial functions $Q_{D,k}(p)$ and $\bar{Q}_{D,k}(q)$ and maybe the number of levels of hierarchy involved in the construction of recurrent formulas of type (41) and (52), should be expected. For example, one can consider two black cubes to be connected iff the minimum path from one to the other contains no more than R_0 white cubes. This would correspond, in some sense, to defects with "spheres of influence," which were investigated in one-scale models[3] ($R_0=0$ corresponds to face-to-face connectedness, i.e., the present paper). With this modification, the problem will have an additional parameter R_0 and, presumably, nontrivial dependence on that parameter. One can consider a new problem by taking the other parameters of the model fixed and letting R_0 vary. It is clear that even if there were no percolation of this "sphere of influence" for small R_0, it would presumably occur for sufficiently large "radius of sphere" R_0, and therefore we would have a percolation phase transition at some intermediate value of R_0.

However, the consideration of more than two levels of hierarchy appears to be necessary already for $R_0=0$ if we consider the $k=2$ case. Here we present the analogs of Eqs. (47) and (54) for the $k=2$ case. Because at the first step of breaking up there are no nontrivial SD (and SC) configurations, one has to consider two steps of breaking up (and therefore three levels of hierarchy) in order to derive the corresponding integral operator. The form of this integral operator is given by

$$
\begin{aligned}
F^{(2)}&[\mathsf{P}\{X\}] \\
&= \rho(X) + [1-\rho(X)] \int dY\, K(X, Y) \Big(\rho^4(Y) \\
&\quad + \rho^3(Y)[1-\rho(Y)] \int dZ\, K(Y, Z)\, Q_{2,2}^{(1)}(\mathsf{P}\{Z\}) \\
&\quad + \rho^2(Y)[1-\rho(Y)]^2 \int dZ\, dU\, K(Y, Z)\, K(Y, U)\, Q_{2,2}^{(2)}(\mathsf{P}\{Z\}, \mathsf{P}\{U\}) \\
&\quad + \rho(Y)[1-\rho(Y)]^3 \int dZ\, dU\, dV\, K(Y, Z)\, K(Y, U)\, K(Y, V) \\
&\quad \times Q_{2,2}^{(3)}(\mathsf{P}\{Z\}, \mathsf{P}\{U\}, \mathsf{P}\{V\}) \\
&\quad + [1-\rho(Y)]^4 \int dZ\, dU\, dV\, dW\, K(Y, Z)\, K(Y, U)\, K(Y, V)\, K(Y, W) \\
&\quad \times Q_{2,2}^{(4)}(\mathsf{P}\{Z\}, \mathsf{P}\{U\}, \mathsf{P}\{V\}, \mathsf{P}\{W\}) \Big)
\end{aligned}
\tag{73}
$$

where the combinatorial functions $Q_{2,2}^{(1)}(p_1)$, $Q_{2,2}^{(2)}(p_1, p_2)$, $Q_{2,2}^{(3)}(p_1, p_2, p_3)$, and $Q_{2,2}^{(4)}(p_1, p_2, p_3, p_4)$ are introduced which count the "strongly defective" configurations of black cubes of two subsequent levels $L+1$ and $L+2$ of the hierarchy within the "parent" cube of level L for the cases when only one (two, three, or four, respectively) of the cubes of level $L+1$ is broken up to cubes of level $L+2$ [and the first term in the integrand takes into account the only SD configuration when none of $(L+1)$-level cubes is broken up]. Their expressions as polynomials of p_i are too involved and do not seem to contain any clarifying information, so we do not produce them here. Equations (48)–(51) are implemented with $F^{(2)}[\cdot]$ instead of $F[\cdot]$. Analogously, for the necessary condition we should use in (56) the following expression instead of $\Phi[\cdot]$ [see (54)]

$$\begin{aligned}
&\Phi^{(2)}[\mathsf{P}\{X\}] \\
&\quad = [1-\rho(X)] \int dY\, K(X, Y) \Big([1-\rho(Y)]^4 \\
&\qquad \times \int dZ\, dU\, dV\, dW\, K(Y, Z)\, K(Y, U)\, K(Y, V)\, K(Y, W) \\
&\qquad \times \bar{Q}_{2,2}^{(4)}(\mathsf{P}\{Z\}, \mathsf{P}\{U\}, \mathsf{P}\{V\}, \mathsf{P}\{W\}) \Big)
\end{aligned} \tag{74}$$

with obvious notations.

Second, one can generalize the branching diffusion part of the problem. For instance, the number r can be considered to be random (possibly as a function of X_t). This, of course, will make the problem of the definition of "strongly defective" and "strictly closing" configurations more complicated because in this case we should deal with face-to-face attached cubes which are broken up in substantially different ways and we should pay much more attention to ensuring the existence of something like Propositions 1 and 2. However, we think that the revision might be only technical (taking into account that the branching diffusion part of the problem may be generalized to this case easily[10]) and in physical applications such a modification can always be absorbed by some corrections to the function $n(X)$.

One can improve the evaluation of both necessary and sufficient conditions considering three (or more) steps instead of two. In principle this can be done and one would obtain equations similar to (73) and (74), but computational difficulties increase very fast, making this method hardly applicable.

For applications it is very interesting to develop method of approximate solution of Eqs. (48) and (56). Sometimes approximations which reduce $K(X, Y)$ to a degenerate kernel are not so bad.

Anyway, the investigated problem has physical significance.[8] It may occur that slightly complicated models of fracture currently used in geophysics[6, 7] as well as some generalizations of the models currently used to describe the intermittent behavior of high-energy scattering processes[11] also will lead to similar problems. We hope that the applications will not be limited to these known areas.

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